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The Digital Form of Operators on Band-Limited Functions

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INTRODUCTION

In signal processing by digital means signals must be represented as periodically abstracted sequences of function values. In many cases a reasonable approximation of the facts, which allows great economy in representation, is the assumption that the signal s is band-limited. This means that the signal has finite total power,

$$\int_{-\infty}^{\infty} |s(t)|^2 \, dt < \infty, \tag{1}$$

and that its spectrum is limited to some finite set Ω of frequencies, i.e.,

$$s(t) = \int_{\Omega} e^{2\pi i f t} g(f) \, df. \tag{2}$$

We shall at first be concerned only with the symmetric interval

$$\Omega = \{f \mid -\tfrac{1}{2} \leqslant f \leqslant \tfrac{1}{2}\}.$$

The set of functions that satisfy Condition (1), if integration is in the Lebesgue sense (functions are considered equivalent if they differ on no more than a set of measure zero), is a complete Hilbert space $L_2(-\infty, \infty)$. The subset of elements in $L_2(-\infty, \infty)$ that satisfy Condition (2) is a subspace which we shall call B. The norm of an element s of B is, of course, given by

$$\| s \| = \left(\int_{-\infty}^{\infty} |s(t)|^2 \, dt \right)^{1/2} = \left(\int_{-1/2}^{1/2} |g(f)|^2 \, df \right)^{1/2}.$$

It is well known that for each $f \in B$ we have the Whittacker–Shannon interpolation formula [1]:

$$f(t) = \lim_{N \to \infty} \sum_{k=-N}^{N} f(n) \frac{\sin \pi(t-k)}{\pi(t-k)}. \tag{3}$$

This result is also known as the Whittacker cardinal series. As we shall see, this formula permits a translation of "continous" operations on f into operations on the set of discrete samples of f, i.e., the values $f(n)$.

The space B while being a subspace of $L_2(-\infty, \infty)$ is free of a property of $L_2(-\infty, \infty)$ that is annoying in applications, namely, that the operations of differentiation and evaluating at a fixed point are unbounded operations. We recall that an operator T on $L_2(-\infty, \infty)$ to $L_2(-\infty, \infty)$ is bounded if there is a number K independent of f such that

$$\int_{-\infty}^{\infty} | Tf |^2 \, dt \leqslant K \cdot \int_{-\infty}^{\infty} |f(t)|^2 \, dt.$$

The norm of T, written $\| T \|$, is the infimum of such number K. A functional H on $L_2(-\infty, \infty)$ which takes its values in the field of complex numbers is bounded if there exists a number K independent of f such that

$$| H(f)| \leqslant K \int_{-\infty}^{\infty} |f(t)|^2 \, dt.$$

The norm of H, written $\| H \|$, is the infimum of such numbers K. The meaning of these notions will be clarified by two examples that demonstrate that the two operations are indeed unbounded in $L_2(-\infty, \infty)$.

EXAMPLE 1. Let

$$f_n(t) = \begin{cases} n & t = 0 \\ 0 & \text{otherwise} \end{cases} \qquad \text{for } n = 1, 2, \ldots.$$

$$f_n(0) = n > \int_{-\infty}^{\infty} |f_n(t)|^2 \, dt = 0.$$

EXAMPLE 2. Let

$$f_n(t) = \begin{cases} 1 + \cos n\pi t & | t | \leqslant 1 \\ 0 & \text{otherwise.} \end{cases}$$

f_n is differentiable everywhere and

$$f_n'(t) = \begin{cases} -n\pi \sin n\pi t & | t | \leqslant 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\int_{-\infty}^{\infty} |f_n'(t)|^2 \, dt = n^2\pi^2 \int_{-1}^{1} \sin^2 n\pi t \, dt = n^2\pi^2$$

$$= \frac{n^2\pi^2}{3} \int_{-\infty}^{\infty} |f_n(t)|^2 \, dt.$$

Differentiation and Evaluation are Bounded in B

The sequences of functions appearing in Examples 1 and 2 are excluded from B because they are "too rapidly varying." In fact, by a theorem of Paley and Wiener, the functions of B are entire functions of exponential type [2]. This means that they are "too smooth" to allow patchy constructions such as those used in the examples. In fact, both differentiation and evaluation at a point are bounded operations in B.

Theorem 1. *The functional H_a defined on B by*

$$H_a f = f(a)$$

is bounded, with norm 1.

Proof.

$$|f(a)| = \left| \int_{-1/2}^{1/2} e^{2\pi i \nu a} g(\nu)\, d\nu \right| \leqslant \left\{ \int_{-1/2}^{1/2} |g(\nu)|^2\, d\nu \right\}^{1/2} = \| f \| .$$

Theorem 2. *The operator D which associates with each function $f \in B$ its derivative is a bounded operator with norm π.*

Proof. Suppose

$$f(t) = \int_{-1/2}^{1/2} e^{2\pi i \nu t} g(\nu)\, d\nu.$$

Then, since the derivative exists,

$$f'(t) = \int_{-1/2}^{1/2} 2\pi i \nu e^{2\pi i \nu t} g(\nu)\, d\nu.$$

Therefore, by Parseval's theorem,

$$\| f' \|^2 = 4\pi^2 \int_{-1/2}^{1/2} |\, \nu\, |^2\, | g(\nu)|^2\, d\nu \leqslant \pi^2 \int_{-1/2}^{1/2} | g(\nu)|^2\, d\nu = \pi^2 \| f \|^2.$$

That is,

$$\| Df \| \leqslant \pi \| f \| .$$

Term by Term Operation on the Interpolation Formula

The aim of this paper is to derive some formulas, useful in digital computations, that allow "continuous" operations to be replaced by "discrete" ones. To this end we must first state Theorem 3 which shows that such interchanges are permissible for bounded operators.

THEOREM 3. *If T is a bounded linear operator on B, then*

$$(Tf)(t) = \lim_{N\to\infty} \sum_{k=-N}^{N} f(k)\,(TS_k)(t) = \lim_{N\to\infty} \sum_{k=-N}^{N} (Tf)(k)\,S_k(t),$$

where

$$S_k(t) = \frac{\sin \pi(t-k)}{\pi(t-k)}.$$

Proof. f being in B, let g be its Fourier transform. Let $\epsilon > 0$. Since $e_n(\nu) = \exp(-2\pi i n\nu)$ $(n = 0, \pm 1, \pm 2,...)$ is a complete system of orthonormal functions for $L_2(-\frac{1}{2}, \frac{1}{2})$ there exists an N such that

$$\left\| g - \sum_{-N}^{N} c_n e_n \right\| < \epsilon/\| T \|,$$

where the norm is taken in $L_2(-\frac{1}{2}, \frac{1}{2})$ and c_n denotes the nth Fourier coefficient of g,

$$c_n = \sum_{-1/2}^{1/2} e^{2\pi i \nu n} g(\nu)\, d\nu = f(n),$$

and $\| T \|$ is the norm of T.

It follows by Parseval's theorem that

$$\left\| f - \sum_{k=-N}^{N} f(k)\, S_k \right\| < \epsilon/\| T \|.$$

Since, by Theorem 1, the functional H_t that yields the value of f at t, i.e.,

$$H_t(f) = f(t),$$

is a functional bounded by 1, we have

$$\left| H_t(Tf) - \sum_{k=-N}^{N} f(k)\, H_t(TS_k) \right|$$
$$\leqslant \left\| Tf - \sum_{k=-N}^{N} f(k)\, TS_k \right\| \leqslant \| T \| \cdot \left\| f - \sum_{k=-N}^{N} f(n)\, S_k \right\| \leqslant \| T \| \cdot \epsilon/\| T \|$$
$$= \epsilon.$$

For the proof of the remaining equality it suffices to note that by Eq. (3) since $Tf \in B$

$$(Tf)(t) = \lim_{N\to\infty} \sum_{k=-N}^{N} (Tf)(k)\, S_k(t).$$

A Numerical Differentiation Formula

An application of Theorems 2 and 3 is the following formula for numerical differentiation.

THEOREM 4. *The samples of the derivative of f are related to the samples of f by the formula*

$$f'(m) = \lim_{N\to\infty} \sum_{\substack{k=m-N \\ k\neq 0}}^{m+N} f(m-k)\,(-1)^k/k. \tag{4}$$

It should be pointed out that it is very important to truncate the formula symmetrically about the mth sample. In tests it has been found that when the series is truncated at $N = 15$, the error is less than 0.1 percent.

Proof. According to Theorem 3,

$$\lim_{N\to\infty} \sum_{k=-N}^{N} f(k)\cdot\pi \left\{ \frac{\pi(t-k)\cos\pi(t-k) - \sin\pi(t-k)}{[\pi(t-k)]^2} \right\}$$
$$= \lim_{N\to\infty} \sum_{k=-N}^{N} f'(k)\,\frac{\sin\pi(t-k)}{\pi(t-k)}\,.$$

Letting $t = m$, we have the desired formula, since the right member vanishes except for the term corresponding to $k = m$.

Similar formulas can be obtained for derivatives of arbitrary order. Let

$$S(x) = \sin \pi x/\pi x = \int_{-1/2}^{1/2} e^{2\pi i\nu x}\, d\nu.$$

Then

$$\frac{d^n S}{dx^n} = \int_{-1/2}^{1/2} (2\pi i\nu)^n\, e^{2\pi i\nu x}\, d\nu. \tag{5}$$

By successive differentiation we find that

$$S^{(m)}(x) = \pi^m \sum_{k=0}^{m} \binom{m}{k} (-1)^k\, k!/(\pi x)^k$$
$$\times \{\alpha_0(k)\sin\pi x + \alpha_1(k)\cos\pi x - \alpha_2(k)\sin\pi x - \alpha_3(k)\cos\pi x\} \tag{6}$$

for $x \neq 0$.

Here

$$\alpha_i(k) = \begin{cases} 1 & \text{if} \quad k = i \bmod 4, \\ 0 & \quad\quad \text{otherwise.} \end{cases}$$

For $x = 0$, we find from Eq. (5) that

$$S^{(m)}(0) = \frac{\pi^m i^m[1 - (-1)^{m+1}]/2}{(m+1)},$$

or

$$S^{(m)}(0) = \frac{\pi^m\{\alpha_0(m) - \alpha_2(m)\}}{(m+1)}. \tag{7}$$

Thus, we can state Theorem 5.

THEOREM 5. *The samples of higher order derivatives of a band-limited function f are related to the samples of f by the formulas*

$$f^{(m)}(k) = \lim_{N\to\infty} \sum_{n=-N}^{N} f(n)\, S^{(m)}(k-n), \tag{8}$$

where $S^{(m)}(x)$ is given by Formulas (6) *and* (7).

PASS BAND SAMPLING

It may happen that the spectrum of a signal lies in the interval $W_0 \leqslant \nu \leqslant W_0 + W$. It has been shown by A. Kohlenberg [3] that we can then write

$$f(t) = \lim_{N\to\infty} \sum_{k=-N}^{N} f(an) \frac{\sin 2\pi(W_0 + W)(t - ak) - \sin 2\pi W_0(t - ak)}{\frac{\pi}{a}(t - ak)}, \tag{9}$$

where a, the sample interval, must satisfy

$$m/2W_0 \leqslant a \leqslant (m+1)/2(W_0 + W)$$

with m being any nonnegative integer. Since the Hilbert space of functions $B(\Omega)$ whose Fourier transforms vanish outside the two symmetrically placed intervals designated by $\Omega = \{\nu \mid W_0 \leqslant |\nu| \leqslant W_0 + W\}$ is simply the direct product of two Hilbert spaces that are isomorphic to B, it is clear that differentiation and evaluation are bounded on $B(\Omega)$. Similarly, Theorem 3 generalizes immediately. The only task remaining is to derive differentiation formulas.

Let

$$S(t) = \frac{\sin 2\pi(W_0 + W)\,t - \sin 2\pi W_0 t}{(\pi/a)\,t}.$$

It is clear that

$$S(t) = a \int_{\Omega} e^{2\pi i\nu t}\, d\nu.$$

Repeated differentiation yields for $t \neq 0$,

$$\begin{aligned} S^{(m)}(t) = \frac{a}{\pi} \sum_{k=0}^{m} \binom{m}{k} \frac{(-1)^{m-k}}{t^{m-k+1}} &[(\alpha_0(k) - \alpha^2(k))\, \{[2\pi(w + w_0)]^k \\ &\times \sin 2\pi(w + w_0)\,t - (2\pi w_0)^k \sin 2\pi w_0 t\} \\ &+ (\alpha_1(k) - \alpha_3(k)\, \{[2\pi(w_0 + w)]^k \cos 2\pi(w_0 + w)\,t \\ &- (2\pi w_0)^k \cos 2\pi w_0 t\}]\,, \end{aligned} \tag{10}$$

where

$$\alpha_i(k) = \begin{cases} 1 & \text{if} \quad k = i \bmod 4, \\ 0 & \text{otherwise.} \end{cases}$$

For $t = 0$ we obtain from the formula

$$S^{(m)}(t) = a \int_{\Omega} (2\pi i\nu)^m\, e^{2\pi i\nu t}\, d\nu, \tag{11}$$

the result

$$S^{m}(0) = a \cdot (2\pi)^m\, (-1)^{m/2} \cdot 2\, \frac{(w_0 + w)^{m+1} - w_0{}^m}{m + 1}, \qquad m \text{ even.} \tag{12}$$

We, thus, have

$$f^{(m)}(ak) = \lim_{N\to\infty} \sum_{n=-N}^{N} f(an)\, S^{(m)}(ak - an), \tag{13}$$

where $S^{(m)}$ is given by formulas (11) and (12).

Convolution Operators

Suppose $h \in L_2(-\infty, \infty)$. Let us define the convolution operator T_h on $B(\Omega)$ to $B(\Omega)$;

$$(T_h f)\,(\tau) = \int_{-\infty}^{\infty} h(\tau - t)\, \overline{f(t)}\, dt = \int_{-\infty}^{\infty} h(u)\, \overline{f(\tau - u)}\, du.$$

An application of Parseval's theorem shows that

$$(T_h f)(\tau) = \int_\Omega e^{2\pi i \nu \tau} H(\nu) \overline{F(\nu)}\, d\nu, \tag{14}$$

where H and F are the Fourier transforms of h and f. Hence, an application of Schwartz's inequality yields that

$$\| T_h f \| \leqslant \| h \|_{B(\Omega)} \cdot \| f \|$$

and, thus, that

$$\| T_h \| \leqslant \| h \|_{B(\Omega)} ,$$

where

$$\| h \|^2_{B(\Omega)} = \int_\Omega | H(\nu) |^2 \, d\nu.$$

Letting $F(\nu)$ be the characteristic function of Ω, i.e., $F(\nu) = X_\Omega(\nu)$, we have from Eq. (14) and Parseval's theorem that

$$\| T_h f_\Omega \|^2 = \| h \|^2_{B(\Omega)} ,$$

where f_Ω is the Fourier transform of X_Ω, from which it follows that T_h is a bounded operator with norm equal to $\| h \|_{B(\Omega)}$.

It may be noted from Eq. 12 that the operator T_h uses in its definition only the projection of h on $B(\Omega)$. Hence, without restricting generality, we may assume that $h \in B(\Omega)$.

Now suppose that $f \in B(\Omega)$. Then

$$f(t) = \lim_{N\to\infty} \sum_{n=-N}^{N} f(na)\, S(t - na),$$

where the specific form of the sampling function S depends on Ω.

Since T_h is bounded we may apply it term by term to $f(t)$, thus, obtaining

$$\begin{aligned} T_h f &= \lim_{N\to\infty} \sum_{n=-N}^{N} f(na)\, T_h S(t - na) \\ &= \lim_{N\to\infty} \sum_{n=-N}^{N} f(na)\, (T_h S)(\tau - na). \end{aligned}$$

But, according to Eq. (14) remembering that $S(\nu)/a$ is the characteristic function of Ω, we have

$$(T_h S)(\tau - na) = ah(\tau - na),$$

and, thus, on setting $\tau = ka$ we have the fundamental Theorem 6.

THEOREM 6. *If $h \in L_2(-\infty, \infty)$ and $f \in B(\Omega)$, the sample values of the convolution of h and f are the discrete convolutions of the sample values of h with the sample values of f. That is*

$$(T_h f)(ka) = \lim_{N\to\infty} a \cdot \sum_{n=-N}^{N} f(na)\, h((k-n)a). \tag{15}$$

By a slight change in notation it follows that the sample values of the cross correlation of two functions is equal to the cross correlation of the sample values.

FOURIER TRANSFORM

Since the Fourier transform is a unitary operator on $L_2(-\infty, \infty)$, and, therefore, also on B we have, as another application of Theorem 3, Theorem 7.

THEOREM 7. *If $f \in B(\Omega)$ and, therefore,*

$$f(t) = \lim_{N\to\infty} \sum_{k=-N}^{N} f(ka)\, S(t-ka),$$

then the Fourier transform g of f is given by

$$g(\nu) = a \cdot \lim_{N\to\infty} \sum_{k=-N}^{N} f(ka)\, e^{-2\pi i \nu k a}$$

for $\nu \in \Omega$ and

$$g(\nu) = 0$$

for $\nu \notin \Omega$.

The importance of this theorem in saving computational labor stands in inverse relation to the difficulty of its proof, which shall be omitted.

THE HILBERT TRANSFORM, AMPLITUDE AND PHASE

The amplitude A and phase φ of a real valued signal s can be defined (see Appendix) in terms of the Hilbert transform $\check{s}$ of s,

$$A(t) = (s^2(t) + \check{s}^2(t))^{1/2} \qquad \text{and} \qquad \varphi(t) = \arc\tan \check{s}(t)/s(t),$$

where

$$\check{s}(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{s(\tau)}{t-\tau}\, d\tau \tag{16}$$

is the Hilbert transform of s. Eq. (16) should be understood in the sense of the Cauchy principal value, i.e.,

$$\check{s}(t) = \frac{1}{\pi} \lim_{\epsilon \to 0} \left(\int_{-\infty}^{t-\epsilon} \frac{s(\tau)}{t-\tau} \, d\tau + \int_{t+\epsilon}^{\infty} \frac{s(\tau)}{t-\tau} \, d\tau \right).$$

Equation (16) defines an operator H on all real-valued functions in B to real-valued functions in B:

$$\check{s} = Rs. \tag{17}$$

It is clear that R is a linear transformation. Let us show that it is also bounded.

In the appendix it is shown that the element of B defined by

$$c = s + i\check{s} = (I + iR)\, s$$

is obtained from s by deleting the negative portion of the spectrum of s. Hence, c is obtained from s by the action of a Hermitian projection operator. Denote this operator by P. Thus,

$$Ps = (I + iR)\, s \qquad \text{and} \qquad R = i(I - P). \tag{18}$$

Since the right member of Eq. (18) is bounded, so is the left member.

It is easily seen by deleting the negative spectrum that the Hilbert transform of

$$s(t) = \frac{\sin \pi t}{\pi t}$$

is

$$\check{s}(t) = \frac{1 - \cos \pi t}{\pi t}$$

and the Hilbert transform of

$$s(t) = \frac{\sin 2\pi(w_0 + w)\, t - \sin 2\pi w_0 t}{(\pi/a)\, t}$$

is

$$\check{s}(t) = \frac{\cos 2\pi w_0 t - \cos 2\pi(w_0 + w)\, t}{(\pi/a)\, t}.$$

The kth sample of the Hilbert transform $\check{f}$ of a signal f in the low pass case is, therefore, given by

$$\begin{aligned} \check{f}(k) &= \lim_{N\to\infty} \sum_{n=-N}^{N} f(n) \frac{1 - (-1)^{k-n}}{\pi(k-n)} \\ &= \lim_{N\to\infty} \sum_{l=k-N}^{k+N} f(k-l) \frac{1 - (-1)^{l}}{\pi l}. \end{aligned} \tag{19}$$

In the pass-band case the kth sample of the Hilbert transform $\check{f}$ of a signal f is given by

$$\check{f}(ak) = \lim_{N\to\infty} \sum_{n=-N}^{N} f(an) \frac{\cos 2\pi w_0 a(k-n) - \cos 2\pi(w_0 + w)\, a(k-n)}{(\pi/a)\,(k-n)}. \quad (20)$$

Appendix

The Hilbert Transform and the Amplitude and Phase of a Signal

If $s(t)$ is a real-valued function of a real variable t it may be of interest to derive from it two other functions of time, the amplitude, and the phase. For example, if $s(t) = \cos t$, we can readily associate with $s(t)$ the complex valued function $c(t) = e^{it}$ which has amplitude (i.e., modulus) 1 and phase t. Moreover, the real part of c is s. If we write $s(t) = \frac{1}{2}(e^{it} + e^{-it})$ we notice that $s(t)$ has both positive and negative spectral components while $c(t)/2$ is formed by suppressing the negative spectral components. This is the procedure to be used in general to obtain the amplitude and phase of a band-limited signal.

If $s(t)$ is a band-limited, real-valued signal, then knowledge of the Fourier transform for positive frequencies is sufficient for the reconstruction of $s(t)$. This is apparent from the formula

$$s(t) = \int_{-w}^{w} e^{2\pi i f t} g(f)\, df. \quad (1A)$$

For, if $s(t) = \overline{s(t)}$, then it follows that $\overline{g(f)} = g(-f)$.

Now consider

$$C(z) = \int_{0}^{w} e^{2\pi i f z} g(f)\, df. \quad (2A)$$

It is clear that $C(z)$ is an entire function of exponential type. Moreover,

$$\begin{aligned} |\, C(t + Re^{i\theta})| &\leqslant \int_0^w e^{-2\pi R \sin\theta f}\, |\, g(f)|\, df \\ &\leqslant B \cdot (1 - e^{-2\pi R \sin\theta w})/2\pi R \sin\theta, \end{aligned} \quad (3A)$$

where

$$B = \max_{|f| \leqslant w} |\, g(f)|\, .$$

Now Cauchy's formula states that

$$C(t) = \frac{1}{2\pi i} \oint \frac{C(\tau)\, d\tau}{\tau - t},$$

where the integral is taken around the curve sketched in Fig. 1.

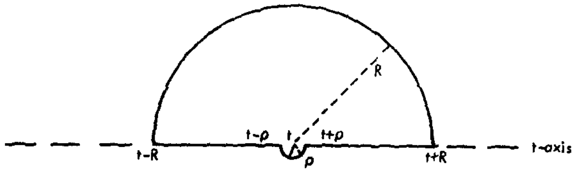

FIG. 1. Contour of integration.

Taking the integrals separately along the components of the curve we have

$$C(t) = \frac{1}{2\pi i}\int_{t-R}^{t-\rho} \frac{C(\tau)\,d\tau}{\tau - t} + \frac{1}{2\pi i}\int_{t+\rho}^{t+R} \frac{C(\tau)\,d\tau}{\tau - t}$$
$$+ \frac{1}{2\pi}\int_0^{\pi} C(t + Re^{i\theta}) + \frac{1}{2\pi}\int_{\pi}^{2\pi} C(t + \rho e^{i\theta})\,d\theta.$$

It is easily seen that

$$\lim_{\rho\to 0} \frac{1}{2\pi}\int_{\pi}^{2\pi} C(t + \rho e^{i\theta})\,d\theta = C(t)/2.$$

Furthermore,

$$\lim_{R\to\infty} \frac{1}{2\pi}\int_0^{\pi} C(t + Re^{i\theta})\,d\theta = 0.$$

Take $\epsilon > 0$. We have that

$$| C(t + Re^{i\theta})| \leqslant W, \qquad 0 \leqslant \theta \leqslant \pi.$$

Let $\delta = \pi\epsilon/(2W)$ and consider

$$\int_0^{\pi} C(t + Re^{i\theta})\,d\theta = \frac{1}{2\pi}\int_0^{\delta} C(t + Re^{i\theta})\,d\theta + \frac{1}{2\pi}\int_{\pi-\delta}^{\delta} C(t + Re^{i\theta})\,d\theta$$
$$+ \frac{1}{2\pi}\int_{\delta}^{\pi-\delta} C(t + Re^{i\theta})\,d\theta.$$

It follows that

$$\int_0^{\pi} C(t + Re^{i\theta})\,d\theta \leqslant \epsilon/2 + \frac{B}{4\pi R \sin\delta}\cdot$$

Now choose R so large that $B/(4\pi R \sin\delta) < \epsilon/2$. Then we have that

$$\int_0^{\pi} C(t + Re^{i\theta})\,d\theta \leqslant \epsilon,$$

and the desired result follows. Cauchy's formula gives then in the limit as $\rho \to 0$ and $R \to \infty$

$$C(t) = \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{C(\tau)\, d\tau}{\tau - t}.$$

Equating real and imaginary parts and writing

$$C(t) = \tfrac{1}{2}(s(t) + i\check{s}(t)),$$

we have

$$s(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\check{s}(\tau)}{\tau - t}\, d\tau \tag{4A}$$

and

$$\check{s}(t) = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{s(\tau)}{\tau - t}\, d\tau. \tag{5A}$$

Formulas (4A) and (5A) express the real and imaginary parts of an "analytic" signal as Hilbert transforms of each other. Using these quantities we can now set down amplitude and phase

$$A(t) = [s(t)^2 + \check{s}^2(t)]^{1/2} \qquad \text{and} \qquad \varphi(t) = \arc\tan \check{s}(t)/s(t).$$

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